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DYNAMIC ANALYSIS OF THE LONG-TERM ZONAL EARTH ENERGY BUDGET EXPERIMENT (LZEEBE) SPACECRAFT

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16. Abstract

An analysis of the motions of the flexible LZEEBE spacecraft after injection into a circular orbit is presented. The spacecraft consists of three spherical balloons connected to a central hub by three long flexible booms which lie in a plane such that the angle between any pair of booms is 120 degrees. The major torques acting on the spacecraft are gravity-gradient torques and torques due to solar radiation pressure acting on the balloons which have different reflective properties. A development of the equations of motion is presented. Computer simulations indicate that the spacecraft will have random motion, provided the injection into orbit does not create conditions favorable to gravity-gradient capture.

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Introduction

This investigation is concerned with the dynamic characteristics of the Long-Term Zonal Earth Energy Budget Experiment (LZEEBE) spacecraft (see Fig. 1) The spacecraft mass center moves in a known circular orbit about the earth. On the other hand, the spacecraft attitude motion is expected to be as random as possible. Hence, capture in a gravity-gradient configuration must be avoided.

The mathematical model consists of a rigid hub and three flexible booms equally spaced in a plane, referred to as the spacecraft plane, where the angle between any pair of booms is 120°. At the tips of the booms there are spherical balloons coated with different materials, so that the solar radiation produces not only forces but also torques.

Assuming that the orbital motion is known, the dynamical problem reduces to that for the rotational motion of the spacecraft as a whole and the elastic motion of the flexible booms. Representing the displacements of the booms by the first bending mode of the associated fixed-base booms with balloons at their tips, the behavior of the spacecraft can be simulated by means of a nine-degree-of-freedom nonlinear system subjected to external excitation. No closed-form solution can be obtained for such a system, so that the equations of motion must be integrated numerically.

The nine-degree-of-freedom simulation has been cast in a form suitable for numerical integration and programmed for digital compuation. Computer results indicate that the spacecraft will perform as desired, provided the injection in orbit does not create conditions favorable to capture.

The Generalized Coordinates

Let us consider an inertial system XYZ with the origin at the sun's center S; axes X and Y are in the ecliptic plane, with axis X along the vernal equinox, and axis Z is normal to the ecliptic plane. The earth's center E moves in the ecliptic plane in a circular orbit around the sun, so that at any time the position of E is defined by the radius vector $\underset{\sim}{R}_E$ making an angle λ_E' with respect to the vernal equinox (see Fig. 2). The equatorial plane of the earth intersects the ecliptic plane along an axis X' parallel to X. Denoting by Y' the axis normal to X' and in the equatorial plane and by Z' the polar axis of the earth, and letting α be the angle of inclination of the polar axis relative to the ecliptic, we conclude that the relation between the systems X'Y'Z' and XYZ can be written in the matrix form

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \left[R(\alpha)\right] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{1}$$

where

$$\left[R(\alpha)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{2}$$

plays the role of a rotation matrix (see Ref. 1).

Next, we wish to define the orbit of the spacecraft. To this end, we introduce the set of axes X"Y"Z" obtained from X'Y'Z' by a rotation λ_N about axis Z'; axis X" defines the ascending node of the spacecraft orbit (see Fig. 3). The relation between X"Y"Z" and X'Y'Z' is simply

$$\begin{Bmatrix} X'' \\ Y'' \\ Z'' \end{Bmatrix} = \left[R(\lambda_N)\right] \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} \tag{3}$$

where

$$\left[R(\lambda_N)\right] = \begin{bmatrix} \cos\lambda_N & \sin\lambda_N & 0 \\ -\sin\lambda_N & \cos\lambda_N & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Denoting by i the orbit inclination relative to the equatorial plane, the spacecraft orbit plane can be defined by axes X"'Y"'Z"' obtained from axes X"Y"Z" by means of the rotation i about X". The relation between the two sets of axes is

$$\begin{Bmatrix} X''' \\ Y''' \\ Z''' \end{Bmatrix} = \left[R(i)\right] \begin{Bmatrix} X'' \\ Y'' \\ Z'' \end{Bmatrix} \tag{5}$$

where

$$\left[R(i)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \tag{6}$$

The spacecraft moves in a circular orbit about E in the X"'Y"' plane. The position in the orbit of the spacecraft mass center C is defined by the radius vector $\underset{\sim}{R}_C$ from E to C, where the direction of the vector is defined by the angle $\psi = \Omega t$ measured from the ascending node. It will prove convenient to introduce a set of "orbital axes" abc with the origin at C and with axis a along $\underset{\sim}{R}_C$, axis b tangent to the orbit and in the direction of motion, and axis c normal to the orbit plane. The relation between systems abc and X"'Y"'Z"' is simply

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \left[R(\psi)\right] \begin{Bmatrix} X''' \\ Y''' \\ Z''' \end{Bmatrix} \tag{7}$$

where

$$\left[R(\psi)\right] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The overall motion of the spacecraft can be conveniently described by a set of body axes xyz, defining the rotational motion of the spacecraft relative to axes abc, and by the elastic deformation of any point on the spacecraft relative to xyz. Assuming that the triad xyz, initially coincident with axes abc, undergoes the rotations θ_2 about y, $-\theta_1$ about x, and θ_3 about z in that order (see Fig. 4), the relation between xyz and abc can be shown to be

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [\ell] \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{9}$$

where $[\ell]$ is the matrix of the direction cosines between the two sets of axes; its explicit form is

$$[\ell] = \begin{bmatrix} c\theta_2\, c\theta_3 - s\theta_1\, s\theta_2\, s\theta_3 & c\theta_1\, s\theta_3 & -s\theta_2\, c\theta_3 - s\theta_1\, c\theta_2\, s\theta_3 \\ -c\theta_2\, s\theta_3 - s\theta_1\, s\theta_2\, c\theta_3 & c\theta_1\, c\theta_3 & s\theta_2\, s\theta_3 - s\theta_1\, c\theta_2\, c\theta_3 \\ c\theta_1\, s\theta_2 & s\theta_1 & c\theta_1\, c\theta_2 \end{bmatrix} \tag{10}$$

where $s\theta_j = \sin\theta_j$ and $c\theta_j = \cos\theta_j$ $(j=1,2,3)$. Note that of all the quantities, introduced to this point, only $\theta_1(t)$, $\theta_2(t)$, and $\theta_3(t)$ are generalized coordinates, as all the remaining quantities are either constant or they are known functions of time.

It follows from the above that the relation between the body axes xyz and the inertial axes XYZ can be written in the compact form

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [L] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{11}$$

where

$$[L] = [\ell][R(\psi)][R(i)][R(\lambda_N)][R(\alpha)] \tag{12}$$

is an overall matrix of direction cosines.

The elastic displacements are assumed to be caused by flexure and to take place in two orthogonal directions. Letting $x_i (i=1,2,3)$ be the nominal longitudinal directions of the booms we shall denote the elastic displacements in the plane of the spacecraft by $v_i(x_i,t)$ and those out of the plane by $w_i(x_i,t)$ (see Fig. 5).

Lagrange's Equations of Motion in General Form

The motion of the spacecraft can be adequately described by nine generalized coordinates, three rotations $\theta_j(t)$ $(j=1,2,3)$ and six elastic displacements $v_i(x_i,t)$ and $w_i(x_i,t)$ $(i=1,2,3)$, where the elastic displacements depend not only on time but also on spatial position. For simplicity, it is assumed that the spacecraft mass center coincides with its geometric center at all times, i.e. its position does not shift relative to the spacecraft because of elastic displacements.

To derive Lagrange's equations, it is necessary to produce expressions for the kinetic energy, potential energy, and work function. First, we must define certain vector quantites. Denoting by $\underset{\sim}{r}_0$ the position vector of any point in the hub relative to C, the position of that point relative to the inertial space is simply

$$\underset{\sim}{R}_0 = \underset{\sim}{R}_E + \underset{\sim}{R}_C + \underset{\sim}{r}_0 \tag{13a}$$

On the other hand, denoting the nominal position of a point on boom i relative to C by $\underset{\sim}{r}_i$ and the elastic displacement of that point by $\underset{\sim}{u}_i$, the position of the point relative to the inertial space is

$$\underset{\sim}{R}_i = \underset{\sim}{R}_E + \underset{\sim}{R}_C + \underset{\sim}{r}_i + \underset{\sim}{u}_i \quad , \qquad i = 1,2,3 \tag{13b}$$

For convenience, let us assume that $\underset{\sim}{R}_E$ and $\underset{\sim}{R}_C$ are expressed in terms of components along the inertial system XYZ and that $\underset{\sim}{r}_o$, $\underset{\sim}{r}_i$, and $\underset{\sim}{u}_i$ are in terms of components along the body axes xyz. If the body axes xyz rotate with the angular velocity $\underset{\sim}{\omega}$ relative to an inertial space, the absolute velocities of the point in question are

$$\dot{\underset{\sim}{R}}_o = \dot{\underset{\sim}{R}}_E + \dot{\underset{\sim}{R}}_C + \underset{\sim}{\omega} \times \underset{\sim}{r}_o \tag{14a}$$

and

$$\dot{\underset{\sim}{R}}_i = \dot{\underset{\sim}{R}}_E + \dot{\underset{\sim}{R}}_C + \dot{\underset{\sim}{u}}_i' + \underset{\sim}{\omega} \times (\underset{\sim}{r}_i + \underset{\sim}{u}_i) \ , \quad i=1,2,3 \tag{14b}$$

where $\dot{\underset{\sim}{u}}_i'$ denotes the elastic velocity of any point on boom i relative to axes xyz.

The kinetic energy can be written in the general form

$$T = \frac{1}{2} \sum_{i=0}^{3} \int_{m_i} \dot{\underset{\sim}{R}}_i \cdot \dot{\underset{\sim}{R}}_i \, dm_i \tag{15}$$

Inserting Eqs. (14) into (15), we obtain

$$T = \frac{1}{2} m(\dot{\underset{\sim}{R}}_E + \dot{\underset{\sim}{R}}_C)\cdot(\dot{\underset{\sim}{R}}_E + \dot{\underset{\sim}{R}}_C) + \frac{1}{2} \sum_{i=0}^{3} \int_{m_i} [\underset{\sim}{\omega} \times (\underset{\sim}{r}_i + \underset{\sim}{u}_i)]\cdot[\underset{\sim}{\omega} \times (\underset{\sim}{r}_i + \underset{\sim}{u}_i)]dm_i$$

$$+ \underset{\sim}{\omega} \cdot \sum_{i=1}^{3} \int_{m_i} (\underset{\sim}{r}_i + \underset{\sim}{u}_i) \times \dot{\underset{\sim}{u}}_i' \, dm_i + \frac{1}{2} \sum_{i=1}^{3} \int_{m_i} \dot{u}_i' \cdot \dot{u}_i' \, dm_i \tag{16}$$

where $m = \Sigma_{i=0}^{3} m_i$ is the total mass of the spacecraft. Moreover, $\Sigma_{i=0}^{3} \int_{m_i} (\underset{\sim}{r}_i + \underset{\sim}{u}_i) dm_i = \underset{\sim}{0}$ by the assumption that the center of mass of the spacecraft does not shift relative to the body axes xyz during motion. Introducing the notation

$$\underset{\sim}{K} = \sum_{i=1}^{3} \int_{m_i} (\underset{\sim}{r}_i + \underset{\sim}{u}_i) \times \dot{\underset{\sim}{u}}_i' \, dm_i \tag{17}$$

where $\underset{\sim}{K}$ is recognized as the angular momentum about C due to elastic velocities

alone, and letting $\underset{\sim}{J}$ be the inertia dyadic of the spacecraft in deformed state about xyz, the kinetic energy can be written in the compact form

$$T = T_C + T_{rel} \tag{18}$$

where

$$T_C = \frac{1}{2} m(\dot{\underset{\sim}{R}}_E + \dot{\underset{\sim}{R}}_C)\cdot(\dot{\underset{\sim}{R}}_E + \dot{\underset{\sim}{R}}_C) \tag{19}$$

is the kinetic energy due to the motion of C and

$$T_{rel} = \frac{1}{2}\underset{\sim}{\omega}\cdot\underset{\sim}{J}\cdot\underset{\sim}{\omega} + \underset{\sim}{\omega}\cdot\underset{\sim}{K} + \frac{1}{2}\sum_{i=1}^{3}\int_{m_i} \dot{\underset{\sim}{u}}_i'\cdot\dot{\underset{\sim}{u}}_i'\, dm_i \tag{20}$$

is the kinetic energy due to the motion of the spacecraft relative to C. Because T_C contains no generalized coordinates or velocities, it will be dropped in future discussions. Consequently, the subscript rel will be dropped in Eq. (20). Equation (20) can also be written in matrix form. Indeed, denoting the inertia matrix representing $\underset{\sim}{J}$ by [J] and the column matrices associated with the vectors $\underset{\sim}{\omega}$, $\underset{\sim}{K}$, and $\dot{\underset{\sim}{u}}_i'$ by $\{\omega\}$, $\{K\}$, and $\{\dot{u}_i'\}$, respectively, Eq. (20) becomes

$$T = \frac{1}{2}\{\omega\}^T[J]\{\omega\} + \{K\}^T\{\omega\} + \sum_{i=1}^{3}\int_{m_i}\{\dot{u}_i'\}^T\{\dot{u}_i'\}\, dm_i \tag{21}$$

At this point it appears desirable to specify some of the quantities in T. It should be pointed out that all these quantities are in terms of components about the body axes xyz. Letting $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$ be unit vectors along x, y, and z, respectively, and recalling that there is a 120° angle between any pair of booms, we can write the position vectors

$$\begin{aligned} \underset{\sim}{r}_1 &= x_1\underset{\sim}{i} \\ \underset{\sim}{r}_2 &= -\frac{1}{2}x_2\underset{\sim}{i} + \frac{\sqrt{3}}{2}x_2\underset{\sim}{j} \\ \underset{\sim}{r}_3 &= -\frac{1}{2}x_3\underset{\sim}{i} - \frac{\sqrt{3}}{2}x_3\underset{\sim}{j} \end{aligned} \tag{22}$$

Moreover, the elastic displacements are as follows:

$$\begin{aligned}
\underset{\sim}{u}_1 &= v_1\underset{\sim}{j} + w_1\underset{\sim}{k} \\
\underset{\sim}{u}_2 &= -\frac{\sqrt{3}}{2} v_2\underset{\sim}{i} - \frac{1}{2} v_2\underset{\sim}{j} + w_2\underset{\sim}{k} \\
\underset{\sim}{u}_3 &= \frac{\sqrt{3}}{2} v_3\underset{\sim}{i} - \frac{1}{2} v_3\underset{\sim}{j} + w_3\underset{\sim}{k}
\end{aligned} \tag{23}$$

so that the relative elastic velocities are simply

$$\begin{aligned}
\dot{\underset{\sim}{u}}_1' &= \dot{v}_1\underset{\sim}{j} + \dot{w}_1\underset{\sim}{k} \\
\dot{\underset{\sim}{u}}_2' &= -\frac{\sqrt{3}}{2} \dot{v}_2\underset{\sim}{i} - \frac{1}{2} \dot{v}_2\underset{\sim}{j} + \dot{w}_2\underset{\sim}{k} \\
\dot{\underset{\sim}{u}}_3' &= \frac{3}{2} \dot{v}_3\underset{\sim}{i} - \frac{1}{2} \dot{v}_3\underset{\sim}{j} + \dot{w}_3\underset{\sim}{k}
\end{aligned} \tag{24}$$

Considering Figs. 2-5, the angular velocity vector can be written in the form

$$\underset{\sim}{\omega} = \omega_x\underset{\sim}{i} + \omega_y\underset{\sim}{j} + \omega_z\underset{\sim}{k} \tag{25}$$

where, assuming that $\dot{\lambda}_N \simeq 0$, the components of $\underset{\sim}{\omega}$ are

$$\begin{aligned}
\omega_x &= -\Omega(s\theta_2\ c\theta_3 + s\theta_1\ c\theta_2\ s\theta_3) - \dot{\theta}_1\ c\theta_3 + \dot{\theta}_2\ c\theta_1\ c\theta_3 \\
\omega_y &= \Omega(s\theta_2\ s\theta_3 - s\theta_1\ c\theta_2\ c\theta_3) + \dot{\theta}_1\ s\theta_3 + \dot{\theta}_2\ c\theta_1\ s\theta_3 \\
\omega_z &= \Omega c\theta_1\ c\theta_2 + \dot{\theta}_2\ s\theta_1 + \dot{\theta}_3
\end{aligned} \tag{26}$$

The moments and products of inertia of the deformed spacecraft are

$$J_{xx} = J_{xxo} + \int_{m_1} (v_1^2 + w_1^2)dm_1 + \int_{m_2} [\tfrac{1}{4}(\sqrt{3}\ x_2 - v_2)^2 + w_2^2]dm_2$$

$$+ \int_{m_3} [\tfrac{1}{4}(\sqrt{3}\ x_3 + v_3)^2 + w_3^2]dm_3$$

$$J_{yy} = J_{yyo} + \int_{m_1} (x_1^2 + w_1^2)dm_1 + \int_{m_2} [\tfrac{1}{4}(x_2 + \sqrt{3}\ v_2)^2 + w_2^2]dm_2$$

$$\int_{m_3} [\tfrac{1}{4}(x_3 - \sqrt{3}\ v_3)^2 + w_3^2]dm_3$$

$$J_{zz} = J_{zzo} + \int_{m_1} (x_1^2 + v_1^2)dm_1 + \frac{1}{4}\int_{m_2} [(x_2 + \sqrt{3}\, v_2)^2 + (\sqrt{3}\, x_2 - v_2)^2]dm_2$$
$$+ \frac{1}{4}\int_{m_3} [(x_3 - \sqrt{3}\, v_3)^2 + (\sqrt{3}\, x_3 + v_3)^2]dm_3$$
$$J_{xy} = J_{xyo} + \int_{m_1} x_1 v_1 dm_1 - \frac{1}{4}\int_{m_2} (x_2 + \sqrt{3}\, v_2)(\sqrt{3}\, x_2 - v_2)dm_2$$
$$+ \frac{1}{4}\int_{m_3} (x_3 - \sqrt{3}\, v_3)(\sqrt{3}\, x_3 + v_3)dm_3 \tag{27}$$
$$J_{xz} = J_{xzo} + \int_{m_1} x_1 w_1 dm_1 - \frac{1}{2}\int_{m_2} (x_2 + \sqrt{3}\, v_2)w_2 dm_2 - \frac{1}{2}\int_{m_3} (x_3 - \sqrt{3}\, v_3)w_3\, dm_3$$
$$J_{yz} = J_{yzo} + \int_{m_1} v_1 w_1 dm_1 + \frac{1}{2}\int_{m_2} (\sqrt{3}\, x_2 - v_2)w_2 dm_2 - \frac{1}{2}\int_{m_3} (\sqrt{3}\, x_3 + v_3)w_3\, dm_3$$

where J_{xxo}, J_{yyo}, ..., J_{yzo} are the moments and products of inertia of the hub. Moreover, the elements of $\{K\}$ have the expressions

$$K_x = \int_{m_1} (v_1\dot{w}_1 - w_1\dot{v}_1)dm_1 + \frac{1}{2}\int_{m_2} [(\sqrt{3}\, x_2 - v_2)\dot{w}_2 - w_2\dot{v}_2]dm_2$$
$$- \frac{1}{2}\int_{m_3} [(\sqrt{3}\, x_3 + v_3)\dot{w}_3 - w_3\dot{v}_3]dm_3$$
$$\tag{28}$$
$$K_y = -\int_{m_1} x_1\dot{w}_1 dm_1 + \frac{1}{2}\int_{m_2} [(x_2 + \sqrt{3}\, v_2)\dot{w}_2 - \sqrt{3}\, w_2\dot{v}_2]dm_2$$
$$+ \frac{1}{2}\int_{m_3} [(x_3 - \sqrt{3}\, v_3)\dot{w}_3 + \sqrt{3}\, w_3\dot{v}_3]dm_3$$
$$K_z = \int_{m_1} x_1\dot{v}_1 dm_1 + \int_{m_2} x_2\dot{v}_2 dm_2 + \int_{m_3} x_3\dot{v}_3 dm_3$$

The potential energy can be written in the general form

$$V = V_{EL} + V_G \tag{29}$$

where V_{EL} is the elastic potential energy having the expression (see Ref. 3)

$$V_{EL} = \frac{1}{2} \sum_{i=1}^{3} \int_{0}^{\ell i} P_{xi} \left[\left(\frac{\partial v_i}{\partial x_i} \right)^2 + \left(\frac{\partial w_i}{\partial x_i} \right)^2 \right] dx_i$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \int_{0}^{\ell i} \left\{ EI_{zi} \left(\frac{\partial^2 v_i}{\partial x_i^2} \right)^2 \left[1 - \frac{5}{2} \left(\frac{\partial v_i}{\partial x_i} \right)^2 \right] + EI_{yi} \left(\frac{\partial^2 w_i}{\partial x_i^2} \right)^2 \left[1 - \frac{5}{2} \left(\frac{\partial w_i}{\partial x_i} \right)^2 \right] \right\} dx_i \tag{30}$$

Note that the first integral in Eq. (30) is due to axial forces and the second is due to flexure. On the other hand, V_G is the gravitational potential energy, which has the matrix form (see Ref. 3)

$$V_G = - \frac{\Omega^2}{2} (\mathrm{tr}\, [J] - 3\{\ell_a\}^T [J] \{\ell_a\}) \tag{31}$$

where $\{\ell_a\}$ is the column matrix of direction cosines between the direction of the vector $\underset{\sim}{R}_C$ (which coincides with the direction of axis a) and axes xyz. Note that a constant term has been ignored in V_G. The matrix $\{\ell_a\}$ has the explicit form

$$\{\ell_a\} = \begin{Bmatrix} \ell_{xa} \\ \ell_{ya} \\ \ell_{za} \end{Bmatrix} = \begin{Bmatrix} c\theta_2\, c\theta_3 - s\theta_1\, s\theta_2\, s\theta_3 \\ -c\theta_2\, s\theta_3 - s\theta_1\, s\theta_2\, c\theta_3 \\ c\theta_1\, s\theta_2 \end{Bmatrix} \tag{32}$$

which is merely the first column of Eq. (10).

It will prove convenient to eliminate the spatial dependence from v_i and w_i ($i=1,2,3$), i.e., to discretize the system. To this end, we use the assumed-modes method (see Ref. 2) and introduce the notation

$$\begin{aligned} v_1(x_1,t) &= \phi_1(x_1)q_1(t), \qquad & w_1(x_1,t) &= \phi_1(x_1)q_2(t) \\ v_2(x_2,t) &= \phi_2(x_2)q_3(t), \qquad & w_2(x_2,t) &= \phi_2(x_2)q_4(t) \\ v_3(x_3,t) &= \phi_3(x_3)q_5(t), \qquad & w_3(x_3,t) &= \phi_3(x_3)q_6(t) \end{aligned} \tag{33}$$

where $\phi_i(x_i)$ (i=1,2,3) are "shape functions". They can be taken as the first bending modes of the booms (see Ref. 2). The functions can be normalized so as to satisfy

$$\int_{m_i} \phi_i^2 \, dm_i = 1 , \qquad i=1,2,3 \tag{34}$$

Moreover, we can introduce the notation

$$\int_{m_i} x_i \phi_i \, dm_i = b_i , \qquad i=1,2,3 \tag{35}$$

and

$$\int_{m_i} x_i^2 \, dm_i = J_i , \qquad i=1,2,3 \tag{36}$$

where J_i is recognized as the moment of inertia of boom i about the z axis. In view of Eqs. (33)-(36), Eqs. (27), (28), and (30) reduce to

$$\begin{aligned}
J_{xx} &= J_{xxo} + \tfrac{3}{4}(J_2+J_3) - \tfrac{\sqrt{3}}{2}(b_2q_3-b_3q_5) + q_1^2 + q_2^2 + \tfrac{1}{4}q_3^2 + q_4^2 + q_5^2 + \tfrac{1}{4}q_6^2 \\
J_{yy} &= J_{yyo} + \tfrac{1}{4}(4J_1+J_2+J_3) + \tfrac{\sqrt{3}}{2}(b_2q_3-b_3q_5) + q_2^2 + \tfrac{3}{4}q_3^2 + q_4^2 + \tfrac{3}{4}q_5^2 + q_6^2 \\
J_{zz} &= J_{zzo} + J_1 + J_2 + J_3 + q_1^2 + q_3^2 + q_5^2 \\
J_{xy} &= J_{xyo} - \tfrac{\sqrt{3}}{4}(J_2-J_3) + b_1q_1 - \tfrac{1}{2}b_2q_3 - \tfrac{1}{2}b_3q_5 + \tfrac{\sqrt{3}}{4}(q_3^2 - q_5^2) \\
J_{xz} &= J_{xzo} + b_1q_2 - \tfrac{1}{2}(b_2q_4 + b_3q_6) - \tfrac{\sqrt{3}}{2}(q_3q_4 - q_5q_6) \\
J_{yz} &= J_{yzo} + \tfrac{\sqrt{3}}{2}(b_2q_4 - b_3q_6) + q_1q_2 - \tfrac{1}{2}(q_3q_4 + q_5q_6)
\end{aligned} \tag{37}$$

$$\begin{aligned}
K_x &= (q_1\dot{q}_2 - q_2\dot{q}_1) + \tfrac{1}{2}[q_4\dot{q}_3 + (\sqrt{3}\, b_2 - q_3)\dot{q}_4 - q_6\dot{q}_5 - (\sqrt{3}\, b_3 - q_5)\dot{q}_6] \\
K_y &= -b_1\dot{q}_2 + \tfrac{1}{2}[\sqrt{3}\, q_4\dot{q}_3 + (b_2 + \sqrt{3}\, q_3)\dot{q}_4 + \sqrt{3}\, q_6\dot{q}_5 + (b_3 - \sqrt{3}\, q_5)\dot{q}_6] \\
K_z &= b_1\dot{q}_1 + b_2\dot{q}_3 + b_3\dot{q}_5
\end{aligned} \tag{38}$$

and

$$V_{EL} = \frac{1}{2}\left\{\left[\int_0^{\ell 1} P_{x1}\left(\frac{d\phi_1}{dx_1}\right)^2 dx_1 + \int_0^{\ell 1} EI_1\left(\frac{d^2\phi_1}{dx_1{}^2}\right)^2 dx_1\right](q_1^2 + q_2^2)\right.$$

$$+ \left[\int_0^{\ell 2} P_{x2}\left(\frac{d\phi_2}{dx_2}\right)^2 dx_2 + \int_0^{\ell 2} EI_2\left(\frac{d^2\phi_2}{dx_2{}^2}\right)^2 dx_2\right](q_3^2 + q_4^2)$$

$$\left.+ \left[\int_0^{\ell 3} P_{x3}\left(\frac{d\phi_3}{dx_3}\right)^2 dx_3 + \int_0^{\ell 3} EI_3\left(\frac{d^2\phi_3}{dx_3{}^2}\right)^2 dx_3\right](q_5^2 + q_6^2)\right\}$$

$$- \frac{5}{2}\left\{\left[\int_0^{\ell 1} EI_1\left(\frac{d^2\phi_1}{dx_1{}^2}\right)^2\left(\frac{d\phi_1}{dx_1}\right)^2 dx_1\right](q_1^4 + q_2^4)\right.$$

$$+ \left[\int_0^{\ell 2} EI_2\left(\frac{d^2\phi_2}{dx_2{}^2}\right)^2\left(\frac{d\phi_2}{dx_2}\right)^2 dx_2\right](q_3^4 + q_4^4)$$

$$\left.+ \left[\int_0^{\ell 3} EI_3\left(\frac{d^2\phi_3}{dx_3{}^2}\right)^2\left(\frac{d\phi_3}{dx_3}\right)^2 dx_3\right](q_5^4 + q_6^4)\right\} \qquad (39)$$

where it was assumed that $EI_{yi} = EI_{zi} = EI_i$ $(i=1,2,3)$. The axial forces are due to centrifugal and gravitational effects and can be obtained from Ref. 3.

Lagrange's equations of motion can be written in the general form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_j}\right) - \frac{\partial T}{\partial \theta_j} + \frac{\partial V}{\partial \theta_j} = \Theta_j \; , \quad j=1,2,3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \; , \quad i=1,2,\ldots,6 \qquad (40)$$

where Θ_j and Q_i are nonconservative generalized forces. In this particular case, they arise from solar radiation and internal damping. Equations (40) are second-order nonlinear differential equations and they are not very convenient for integration purposes; it is actually more convenient to work with first-order differential equations. Our efforts will be directed next toward producing such a set of equations.

First-Order Differential Equations

Considering Eq. (21) and recognizing that $\sum_{i=1}^{3} \int_{m_i} \{\dot{u}_i'\}^T\{\dot{u}_i'\}dm_i = \sum_{i=1}^{6} \dot{q}_i^2$, the second set of Eqs. (40) can be written in the form

$$\left\{ \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right)\right\}^T\{\omega\} + \left\{\frac{\partial K}{\partial \dot{q}_i}\right\}^T\{\dot{\omega}\} + \ddot{q}_i - \frac{1}{2}\{\omega\}^T\left[\frac{\partial J}{\partial q_i}\right]\{\omega\}$$

$$-\left\{\frac{\partial K}{\partial q_i}\right\}^T\{\omega\} + \frac{\partial V}{\partial q_i} = Q_i , \quad i=1,2,\ldots,6 \tag{41}$$

To transform Eqs. (41) into a set of twelve first-order equations, let us introduce the auxiliary variables

$$p_i = \dot{q}_i \ , \quad i = 1,2,\ldots,6 \tag{42}$$

In addition, let us introduce the 3 x 6 matrix [K*] defined as

$$[K^*] = \begin{bmatrix} \frac{\partial K_x}{\partial \dot{q}_1} & \frac{\partial K_x}{\partial \dot{q}_2} & --- & \frac{\partial K_x}{\partial \dot{q}_6} \\ \frac{\partial K_y}{\partial \dot{q}_1} & \frac{\partial K_y}{\partial \dot{q}_2} & --- & \frac{\partial K_y}{\partial \dot{q}_6} \\ \frac{\partial K_z}{\partial \dot{q}_1} & \frac{\partial K_z}{\partial \dot{q}_2} & --- & \frac{\partial K_z}{\partial \dot{q}_6} \end{bmatrix} \tag{43}$$

as well as the modified generalized forces

$$Q_i^* = Q_i - \left\{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i}\right\}^T\{\omega\} + \frac{1}{2}\{\omega\}^T\left[\frac{\partial J}{\partial q_i}\right]\{\omega\} - \frac{\partial V}{\partial q_i}$$

$$i=1,2,\ldots,6 \tag{44}$$

so that, arranging p_i and Q_i^* in column matrices, we can write Eqs. (41) and (42) in the form of the first-order differential equations

$$\{\dot{p}\} + [K^*]^T\{\dot{\omega}\} = \{Q^*\} \tag{45a}$$

$$\{\dot{q}\} = \{p\}$$

Next, let us turn our attention to the first set of Eqs. (40). The equations are in terms of the actual angular coordinates θ_j. It will prove more convenient, however, to work with a set of equations in terms of quasi-coordinates (Ref. 1). These equations can be written in the form

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right)\right\} + [\omega]\left\{\frac{\partial T}{\partial \omega}\right\} = \{N\}_G + \{N\}_R \tag{46}$$

where $[\omega]$ is the skew symmetric matrix of angular velocity components, i.e.

$$[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{47}$$

and $\{N\}_G$ and $\{N\}_R$ are the gravitational and radiation pressure torques, respectively. The gravitational torque can be shown to have the components (see Ref. 1, p. 437)

$$\begin{aligned} N_{xG} &= 3\Omega^2[(J_{zz}-J_{yy})\ell_{ya}\ell_{za} + J_{xy}\ell_{xa}\ell_{za} - J_{xz}\ell_{xa}\ell_{ya} + J_{yz}(\ell_{za}^2 - \ell_{ya}^2)] \\ N_{yG} &= 3\Omega^2[(J_{xx}-J_{zz})\ell_{xa}\ell_{za} + J_{yz}\ell_{xa}\ell_{ya} - J_{xy}\ell_{ya}\ell_{za} + J_{xz}(\ell_{xa}^2 - \ell_{za}^2)] \\ N_{zG} &= 3\Omega^2[(J_{yy}-J_{xx})\ell_{xa}\ell_{ya} + J_{xa}\ell_{ya}\ell_{za} - J_{yz}\ell_{xa}\ell_{za} + J_{xy}(\ell_{ya}^2 - \ell_{xa}^2)] \end{aligned} \tag{48}$$

where ℓ_{xa}, ℓ_{ya}, ℓ_{za} are given by Eq. (32). The radiation pressure torque is derived in the next section.

From Eq. (21), we conclude that

$$\left\{\frac{\partial T}{\partial \omega}\right\} = [J]\{\omega\} + \{K\} \tag{49}$$

so that, introducing Eq. (49) into Eq. (46), we obtain

$$[\dot{J}]\{\omega\} + [J]\{\dot{\omega}\} + \{\dot{K}\} + [\omega][J]\{\omega\} + [\omega]\{K\} = \{N\}_G + \{N\}_R \tag{50}$$

Next, we observe that

$$\{\dot{K}\} = [K^*]\{\dot{p}\} \tag{51}$$

so that, introducing the notation

$$\{N^*\} = \{N\}_G + \{N\}_R - [\dot{J}]\{\omega\} - [\omega]([J]\{\omega\} + \{K\}) \tag{52}$$

Eq. (50) reduces to

$$[K^*]\{\dot{p}\} + [J]\{\dot{\omega}\} = \{N^*\} \tag{53}$$

On the other hand, from Eqs. (26), we conclude that the angular velocity vector can be written in the matrix form

$$\{\omega\} = [\theta]\{\dot{\theta}\} + \Omega\{\ell_c\} \tag{54}$$

where

$$[\theta] = \begin{bmatrix} -c\theta_3 & c\theta_1\ s\theta_3 & 0 \\ s\theta_3 & c\theta_1\ c\theta_3 & 0 \\ 0 & s\theta_1 & 1 \end{bmatrix} \tag{55}$$

and

$$\{\ell_c\} = \begin{Bmatrix} -(s\theta_2\ c\theta_3 + s\theta_1\ c\theta_2\ s\theta_3) \\ s\theta_2\ s\theta_3 - s\theta_1\ c\theta_2\ c\theta_3 \\ c\theta_1\ c\theta_2 \end{Bmatrix} \tag{56}$$

where $\{\ell_c\}$ is recognized as the matrix of direction cosines between axis c and the body axes xyz (see the last column of Eq. (10)). Equation (54) can be rewritten in the form

$$\{\dot{\theta}\} = [\theta]^{-1}(\{\omega\} - \Omega\{\ell_c\}) \tag{57}$$

Equations (53) and (57) represents another two sets of first-order differential equations replacing the set (46).

Next, let us introduce the state vector

$$\{x\} = \left\{ \begin{array}{c} \{p\} \\ \hline \{\omega\} \\ \hline \{q\} \\ \hline \{\theta\} \end{array} \right\} \tag{58}$$

which has the dimension eighteen. Then introducing the square matrix

$$[A] = \left[\begin{array}{c:c:c:c} [1] & [K^*]^T & [0] & [0] \\ \hdashline [K^*] & [J] & [0] & [0] \\ \hdashline [0] & [0] & [1] & [0] \\ \hdashline [0] & [0] & [0] & [1] \end{array} \right] \tag{59}$$

as well as the column matrix of "generalized forces"

$$\{X\} = \left\{ \begin{array}{c} \{Q^*\} \\ \hdashline \{N^*\} \\ \hdashline \{p\} \\ \hdashline [\theta]^{-1}(\{\omega\} - \Omega\{\ell_c\}) \end{array} \right\} \tag{60}$$

Eqs. (45), (53), and (57) can be written in the compact form

$$[A]\{\dot{x}\} = \{X\} \tag{61}$$

where the elements of $\{X\}$ are generally nonlinear functions of the state vector. Equation (61) can be rewritten as

$$\{\dot{x}\} = [A]^{-1}\{X\} \tag{62}$$

which represents a form suitable for numerical integration. Note that Eq. (62) implies the existence of $[A]^{-1}$, which in turn implies the existence of $[\theta]^{-1}$, with the further implication that $[\theta]$ is nonsingular. From Eq. (55), we conclude that det $[\theta] \neq 0$ provided $\cos\theta_1 \neq 0$. Hence, at values of θ_1 that are odd integer multiples of $\pi/2$ Eq. (62) cannot be used, so that such values must be avoided in a numerical solution.

Radiation Forces and Torques

There are several sources of radiation that can cause forces and possible torques on the spacecraft. These sources are:

1. Direct solar photon radiation
2. Solar radiation reflected by the earth and its atmosphere
3. Radiation from the earth and its atmosphere
4. Radiation from the spacecraft.

The last three are one order of magnitude smaller than the first, so that they will be ignored. Moreover, it will be assumed that the solar photon radiation on the booms is negligible. Hence, the only forces and torques to be considered are caused by solar photon radiation on the balloons.

Next, we wish to obtain an expression for the force vector on a typical balloon. To this end, let us consider a differential element of area dA (see Fig. 6) and denote by $\underset{\sim}{n}$ the unit vector normal to the surface and pointing outward and by $\underset{\sim}{s}$ the unit vector tangent to the surface and in the direction shown. Then, letting θ be the angle between the incident radiation and $\underset{\sim}{n}$, the photon radiation force corresponding to the area dA is (see Ref. 4)

$$d\underset{\sim}{F} = \frac{I}{c}\left[-\left[(1+c_{rs})\cos\theta + \frac{2}{3}c_{rd}\right]\underset{\sim}{n} + (1-c_{rs})\sin\theta\ \underset{\sim}{s}\right]\cos\theta\ dA \tag{63}$$

where I = energy per unit time through a unit cross sectional-area (in W/m^2)

c = speed of light

c_{rs} = coefficient of specular reflection

c_{rd} = coefficient of diffuse reflection

To obtain the force exerted on the balloon, we must integrate Eq. (63) over the area, which is assumed to be spherical. From Fig. 6, we can write

$$\begin{aligned} \underset{\sim}{n} &= \sin\theta\cos\phi\ \underset{\sim}{1}_\xi + \sin\theta\sin\phi\ \underset{\sim}{1}_\eta + \cos\theta\ \underset{\sim}{1}_\zeta \\ \underset{\sim}{s} &= \cos\theta\cos\phi\ \underset{\sim}{1}_\xi + \cos\theta\sin\phi\ \underset{\sim}{1}_\eta - \sin\theta\ \underset{\sim}{1}_\zeta \end{aligned} \tag{64}$$

where $\underset{\sim}{1}_\xi$, $\underset{\sim}{1}_\eta$, $\underset{\sim}{1}_\zeta$ are unit vectors along axes ξ, η, ζ, respectively. Moreover, the differential element of area has the expression

$$dA = r^2 \sin\theta \, d\theta \, d\phi \tag{65}$$

where r is the radius of the balloon. Denoting the force vector by

$$\underset{\sim}{F} = F_\xi \underset{\sim}{1}_\xi + F_\eta \underset{\sim}{1}_\eta + F_\zeta \underset{\sim}{1}_\zeta \tag{66}$$

and integrating over the surface of the balloon, the force components can be shown to be

$$\begin{aligned} F_\xi &= F_\eta = 0 \\ F_\zeta &= -2\pi r^2 \frac{I}{c}\left(\frac{1}{2} + \frac{2}{9} c_{rd}\right) \end{aligned} \tag{67}$$

For further reference, we wish to calculate the forces on the various balloons in terms of components along the spacecraft body axes xyz. In view of Eqs. (67), we conclude that the force on any balloon has the magnitude

$$|\underset{\sim}{F}_i| = F_{i\zeta} = 2\pi r_i^2 \frac{I}{c}\left(\frac{1}{2} + \frac{2}{9} c_{rdi}\right), \qquad i=1,2,3 \tag{68}$$

where allowance has been made for the possibility that the balloons are of different sizes and that the coefficients of diffuse reflection are different. The direction of the force vectors coincides with the direction of the solar radiation. Hence, the direction of $\underset{\sim}{F}_i$ is parallel to the vector $\underset{\sim}{R}_E + \underset{\sim}{R}_C$. Because $|\underset{\sim}{R}_E| \gg |\underset{\sim}{R}_C|$, it will be assumed that the direction is parallel to $\underset{\sim}{R}_E$. This direction can be expressed in terms of inertial components as follows:

$$\frac{\underset{\sim}{R}_E}{|\underset{\sim}{R}_E|} = \cos\lambda_E \, \underset{\sim}{I} + \sin\lambda_E \, \underset{\sim}{J} \tag{69}$$

where $\underset{\sim}{I}$ and $\underset{\sim}{J}$ are unit vectors along the inertial axes X and Y, respectively (see Fig. 2). It follows that

$$\underset{\sim}{F}_i = 2\pi r_i^2 \frac{I}{c} \left(\frac{1}{2} + \frac{2}{9} c_{rdi}\right)(\cos\lambda_E \underset{\sim}{I} + \sin\lambda_E \underset{\sim}{J}) \tag{70}$$

The same vector can be expressed in terms of components along the spacecraft body axes xyz by writing

$$\underset{\sim}{F}_i = F_{ix} \underset{\sim}{i} + F_{iy} \underset{\sim}{j} + F_{iz} \underset{\sim}{k} \tag{71}$$

where the components F_{ix}, F_{iy}, and F_{iz} can be obtained from Eq. (70) by means of the coordinate transformation (11).

Next, we wish to use Eq. (71) and determine the radiation force and torque vectors $\{Q\}_R$ and $\{N\}_R$. Ignoring the radius of the balloons compared to the length of the booms and using Eqs. (22), (23), and (33), we can write the position of the balloons as follows:

$$\begin{aligned}
\underset{\sim}{r}_1(\ell_1) + \underset{\sim}{u}_1(\ell_1,t) &= \ell_1 \underset{\sim}{i} + \phi_1(\ell_1)q_1(t)\underset{\sim}{j} + \phi_1(\ell_1)q_2(t)\underset{\sim}{k} \\
\underset{\sim}{r}_2(\ell_2) + \underset{\sim}{u}_2(\ell_2,t) &= -\frac{1}{2}[\ell_2 + \sqrt{3}\,\phi_2(\ell_2)q_3(t)]\underset{\sim}{i} + \frac{1}{2}[\sqrt{3}\,\ell_2 - \phi_2(\ell_2)q_3(t)]\underset{\sim}{j} \\
&\quad + \phi_2(\ell_2)q_4(t)\underset{\sim}{k} \qquad (72) \\
\underset{\sim}{r}_3(\ell_3) + \underset{\sim}{u}_3(\ell_3,t) &= -\frac{1}{2}[\ell_3 - \sqrt{3}\,\phi_3(\ell_3)q_5(t)]\underset{\sim}{i} - \frac{1}{2}[\sqrt{3}\,\ell_3 + \phi_3(\ell_3)q_t(t)]\underset{\sim}{j} \\
&\quad + \phi_3(\ell_3)q_6(t)\underset{\sim}{k}
\end{aligned}$$

But the virtual work associated with the radiation forces can be written in terms of both actual coordinates and forces and generalized coordinates and forces in the form

$$\overline{\delta W}_R = \sum_{i=1}^{3} \underset{\sim}{F}_i \cdot \delta\underset{\sim}{u}_i(\ell_i,t) = \sum_{i=1}^{6} Q_{iR}\,\delta q_i \tag{73}$$

Inserting Eqs. (71) and (72) into (73) and equating coefficients of δq_i $(i=1,2,\ldots,6)$, we obtain the generalized force components

$$
\begin{aligned}
Q_{1R} &= F_{1y}\,\phi_1(\ell_1) \\
Q_{2R} &= F_{1z}\,\phi_1(\ell_1) \\
Q_{3R} &= -\frac{1}{2}\,(\sqrt{3}\,F_{2x} + F_{2y})\phi_2(\ell_2) \\
Q_{4R} &= F_{2z}\,\phi_2(\ell_2) \\
Q_{5R} &= \frac{1}{2}\,(\sqrt{3}\,F_{3x} - F_{3y})\phi_3(\ell_3) \\
Q_{6R} &= F_{3z}\,\phi_3(\ell_3)
\end{aligned}
\tag{74}
$$

The radiation torque is obtained by writing simply

$$
\underset{\sim}{N}_R = \sum_{i=1}^{3} [\underset{\sim}{r}_i(\ell_i) + \underset{\sim}{u}_i(\ell_i,t)] \times \underset{\sim}{F}_i \tag{75}
$$

Inserting Eqs. (71) and (72) into (75) we obtain the components

$$
\begin{aligned}
N_{xR} &= (F_{1z}q_1 - F_{1y}q_2)\phi_1(\ell_1) - (\frac{1}{2}\,F_{2z}q_3 + F_{2y}q_4)\phi_2(\ell_2) \\
&\quad - (\frac{1}{2}\,F_{3z}q_5 + F_{3y}q_6)\phi_3(\ell_3) + \frac{\sqrt{3}}{2}\,(F_{2z}\ell_2 - F_{3z}\ell_3) \\
N_{yR} &= F_{1x}q_2\phi_1(\ell_1) + (\frac{\sqrt{3}}{2}\,F_{2z}q_3 + F_{2x}q_4)\phi_2(\ell_2) - (\frac{\sqrt{3}}{2}\,F_{3z}q_5 - F_{3x}q_6)\phi_3(\ell_3) \\
&\quad + \frac{1}{2}\,(F_{2z}\ell_2 + F_{3z}\ell_3) - F_{1z}\ell_1 \\
N_{zR} &= -F_{1x}q_1\phi_1(\ell_1) + \frac{1}{2}\,(F_{2x} - \sqrt{3}\,F_{2y})q_3\phi_2(\ell_2) + \frac{1}{2}\,(F_{3x} + \sqrt{3}\,F_{3y})q_5\phi_3(\ell_3) \\
&\quad + F_{1y}\ell_1 - \frac{1}{2}(\sqrt{3}\,F_{2x} + F_{2y})\ell_2 + \frac{1}{2}\,(\sqrt{3}\,F_{3x} - F_{3y})\ell_3
\end{aligned}
\tag{76}
$$

It remains to determine when these radiation effects are present, i.e., when solar radiation impinges on the spacecraft. This coincides with the period when the spacecraft is illuminated by the sun. For simplicity we shall be concerned only with shadowing of the spacecraft by the earth.

Let us denote the radius of the earth by r_E and the angle between $\underset{\sim}{R}_E$ and $\underset{\sim}{R}_C$ by γ (see Fig. 7). Once again we assume that $|\underset{\sim}{R}_E| \gg |\underset{\sim}{R}_C|$. But from the definitions of the cross and dot products of vectors, we can write

$$\sin \gamma = \frac{|\underset{\sim}{R}_E \times \underset{\sim}{R}_C|}{|\underset{\sim}{R}_E| \; |\underset{\sim}{R}_C|} \quad , \quad \cos\gamma = \frac{\underset{\sim}{R}_E \cdot \underset{\sim}{R}_C}{|\underset{\sim}{R}_E| \; |\underset{\sim}{R}_C|} \tag{77}$$

Because solar radiation is in the direction of $\underset{\sim}{R}_E$, we observe that the spacecraft is in the shadow of the earth when the projection of $\underset{\sim}{R}_C$ onto $\underset{\sim}{R}_E$ is positive and when $\sin\gamma < r_E / |\underset{\sim}{R}_C|$. Hence, radiation forces will not be present when

$$\cos\gamma > 0 \quad \text{and} \quad |\underset{\sim}{R}_E \times \underset{\sim}{R}_C| < r_E \, |\underset{\sim}{R}_E| \tag{78}$$

Criteria (78) can be written in the explicit form

$$\begin{aligned} &\cos\lambda_E (\cos \psi \cos\lambda_N - \sin\psi \cos i \sin \lambda_N) + \sin \lambda_E \cos\alpha (\cos \psi \sin \lambda_N \\ &\quad + \sin \psi \cos i \cos \lambda_N) + \sin \lambda_E \sin \alpha \sin \psi \sin i > 0 \end{aligned} \tag{79a}$$

and

$$\begin{aligned} &\{\sin^2 \lambda_E [\sin \psi (\sin i \cos \alpha - \cos i \sin \alpha \cos \lambda_N) - \cos \psi \sin \alpha \sin \lambda_N]^2 \\ &\quad + [\cos \psi \sin \alpha \sin \lambda_E \cos \lambda_N - \sin \psi (\sin i \cos \lambda_E + \cos i \sin \alpha \sin \lambda_E \sin \lambda_N)]^2 \\ &\quad + [\sin \psi \cos i (\cos \lambda_E \cos \lambda_N + \cos \alpha \sin \lambda_E \sin \lambda_N) \\ &\qquad + \cos \psi (\cos \lambda_E \sin \lambda_N - \sin \lambda_E \cos \alpha \cos \lambda_N)]^2\}^{\frac{1}{2}} < r_E/R_C \end{aligned} \tag{79b}$$

where $R_C = |\underset{\sim}{R}_C|$.

Damping Forces

The other nonconservative forces acting on the system are the internal damping forces. This type of damping is generally known as structural damping, but the forces are often modelled as viscous forces. Of course, the forces being internal they produce no torques.

Letting c_i be the damping coefficient per unit length of boom i, we can introduce the Rayleigh dissipation function in the form (see Ref. 1)

$$F = \frac{1}{2} \sum_{i=1}^{3} \int_0^{\ell i} c_i \left[\left(\frac{\partial v_i}{\partial t} \right)^2 + \left(\frac{\partial w_i}{\partial t} \right)^2 \right] dx_i \tag{80}$$

Using Eqs. (33), Eq. (80) becomes

$$F = \frac{1}{2} [C_1(\dot{q}_1^2 + \dot{q}_2^2) + C_2(\dot{q}_3^2 + \dot{q}_4^2) + C_3(\dot{q}_5^2 + \dot{q}_6^2)] \tag{81}$$

where

$$C_i = \int_0^{\ell i} c_i \phi_i^2 \, dx_i \ , \qquad i=1,2,3 \tag{82}$$

The damping forces can be obtained from Rayleigh's dissipation function by means of the formula

$$Q_{iD} = \frac{\partial F}{\partial \dot{q}_i} \ , \qquad i=1,2,\ldots,6 \tag{83}$$

from which it follows that

$$\begin{aligned} Q_{1D} &= C_1 \dot{q}_1 \ , \quad Q_{2D} = C_1 \dot{q}_2 \ , \quad Q_{3D} = C_2 \dot{q}_3 \\ Q_{4D} &= C_2 \dot{q}_4 \ , \quad Q_{5D} = C_3 \dot{q}_5 \ , \quad Q_{6D} = C_3 \dot{q}_6 \end{aligned} \tag{84}$$

Quite often damping is given in terms of the damping factor ζ_i rather than in terms of the damping coefficient c_i. The relation between the two is

$$2\zeta_i \omega_i = c_i/\rho_i \ , \qquad i=1,2,3 \tag{85}$$

where ω_i is the lowest natural frequency of boom i and ρ_i is the mass per unit length of the boom. Letting the mass of boom and balloon have the expression

$$\rho_i(x_i) = \rho_i + M_i \, \delta(x_i - \ell_i) \ , \qquad i=1,2,3 \tag{86}$$

where ρ_i on the right side is the constant mass per unit length of boom introduced in Eq. (85), M_i is the mass of the balloon, and $\delta(x_i - \ell_i)$ is a spatial Dirac delta function, Eq. (34) yields

$$\int_0^{\ell i} [\rho_i + M_i \ \delta(x_i - \ell_i)]\phi_i^2 \ dx_i = \rho_i \int_0^{\ell i} \phi_i^2 \ dx_i + M_i\phi_i^2(\ell_i) = 1 \tag{87}$$

so that Eq. (82) yields

$$C_i = c_i \frac{1 - M_i\phi_i{}^2(\ell_i)}{\rho_i} = 2\zeta_i\omega_i \ [1 - M_i\phi_i{}^2(\ell_i)] \ , \ i=1,2,3 \tag{88}$$

Equations (88) can be inserted in Eqs. (84) to express the damping forces in terms of the damping factors ζ_i.

Mathematical Solution and Computer Program

The numerical solution of the equations of motion involves several steps, namely, the determination of the lowest natural frequency and mode for each boom, the evaluation of certain definite integrals, the inversion of the matrix [A], Eq. (59), and the numerical integration of Eqs. (62). The computer program accomplishes all of these things with the aid of several subroutines contained in the IBM Scientific Subroutine Package.

The lowest natural frequency for each boom is obtained by solving for the first root of the associated characteristic equation numerically. A simple method of interval halving produces this root. All the necessary definite integrals are evaluated using the trapezoidal rule in conjunction with Romberg's extrapolation method. Two hundred intervals are used in integrating each function. Matrix [A] is inverted by a standard Gauss-Jordan numerical procedure. An approximate solution of the first-order differential equations of motion, Eqs. (62), for given initial conditions, is obtained by a Runge-Kutta integration procedure. Evaluation is done by means of fourth-order Runge-Kutta formulas with the modification by Gill. Accuracy is tested comparing the results of the procedure by using single and double increments. This method is both stable and self starting.

Numerical Results

The dynamics problem of the LZEEBE spacecraft has been programmed for digital computation and several cases of interest have been investigated. These cases appear to be the most significant ones and they all differ in the initial conditions. These initial conditions can be divided broadly into two types: (1) zero (or nearly zero) initial velocities relative to an inertial space and zero (or nearly zero) initial displacements relative to an orbital system of axes, and (2) zero (or nearly zero) initial displacements and velocities relative to an orbital system of axes. These cases are significant because they can shed some light on the possibility that the spacecraft might be captured in a planar gravity-gradient stabilization configuration.

The cases investigated and the results are as follows:

Case 1. Zero initial velocities relative to an inertial space and zero initial displacements relative to the orbital axes.

Some of the parameters have the values: $\lambda_E = 0°$, $\lambda_N = 0°$, $\psi = 0°$, and $i = 60°$. The spacecraft begins its motion in the earth's shadow. Because the initial tendency of the body orientation is to remain fixed in an inertial space, the angle θ_3 tends initially to increase in magnitude at a rate equal to the orbital velocity Ω. This tends to introduce a very small gravity torque about the z axis. At the same time, the differential gravity forces on the booms cause them to deflect, with most of the elastic displacements taking place in the plane of the spacecraft. As soon as the spacecraft emerges from the earth's shadow, the sun's radiation pressure begins to exert forces and torques on the spacecraft, causing the spacecraft to rotate and the booms to deflect. Whereas the boom deflections remain well below one meter, the angles θ_j become large. In particular, the angle θ_2 exceeds 2π,

thus ensuring a complete rotation of the spacecraft about its own center. Because of the ever-changing pattern of motion, damping appears to have no meaningful effect on the elastic displacements.

In view of the large rotations of the spacecraft, it appears that the sun's radiation torques are sufficient to ensure that all sides of the spacecraft are exposed to the sun in a random-like fashion.

For the duration of the computer run, a time period equivalent to about 20 orbits, there was no indication that the spacecraft might settle into a planar gravity-gradient stabilization equilibrium.

Case 2. Zero or small initial angular displacements and velocities with respect to the orbital axes, with booms 2 and 3 deformed by differential gravity and centrifugal forces.

A. $\theta_1 = \theta_2 = \theta_3 = 0$, $\lambda_E = 90°$, $\lambda_N = 180°$, $\psi = 0°$, $i = 66.5°$, $\omega_z = \Omega$.

This case is designed to assess the effect of the sun's radiation pressure on the spacecraft in the initial planar gravity-gradient equilibrium. Indeed, in this case the sun's radiation forces are initially normal to the orbit plane, and hence to the spacecraft plane, and remain constant in magnitude and direction for time intervals of the order of one day. (It should be pointed out that the earth's oblateness causes the orbit plane to precess at a rate of the order of 3° per day, so that, after a while, the radiation pressure ceases to be normal to the orbit plane.) The radiation forces cause booms 2 and 3 to oscillate about the deformed equilibrium with an amplitude of the order of 0.04 meters tip deflection. Small θ_1 and θ_2 oscillations are also present. Hence, the initial planar gravity-gradient stabilization equilibrium is largely maintained. It follows that, when working against the stabilizing effect of differential gravity and centrifugal forces, the radiation pressure has a very small effect.

B. $\theta_1 = \theta_3 = 0, \theta_2 = 5°, \lambda_E = 0°, \lambda_N = 0°, \psi = 0°, i = 60°, \omega_z = \Omega.$

The booms undergo small oscillations about the deformed equilibrium as in the case 2A. In addition, the spacecraft oscillates between $\theta_2 = \pm 5°$ with a period of approximately 3,000 sec. The gyroscopic effect also induces an oscillation about the x axis with the same amplitude but with twice the period. Once again the radiation pressure effect on the equilibrium state is minimal. There appears to be a small secular rate of reduction in ω_z.

Summary and Conclusions

The dynamical behavior of the LZEEBE spacecraft has been investigated under certain simplifying assumptions. In particular, it is assumed that the spacecraft center of mass moves in a circular orbit around the earth, so that its motion in an inertial space is known. The spacecraft is subjected to solar radiation forces and differential-gravity forces. Thermal bending effects have been ignored, an assumption justified when the spacecraft rotates about its own mass center in a way that no one side is exposed continuously to the sun. The formulation consists of the system Lagrange's equations of motion for the three rotations $\theta_j(t)$ $(j=1,2,3)$ of the spacecraft as a whole and for the six elastic displacements $v_i(x_i,t)$ and $w_i(x_i,t)$ $(i=1,2,3)$ of the booms, where v_i and w_i are the displacements of boom i in the plane of the spacecraft and normal to the plane, respectively. Note that the rotations θ_j are measured relative to an orbiting set of axes abc, where a coincides with the local vertical, b is tangent to the orbit and in the direction of the orbital motion, and c is normal to the orbit. The rotations θ_j define the orientation in space of the spacecraft body axes xyz; the elastic displacements are measured relative to axes xyz.

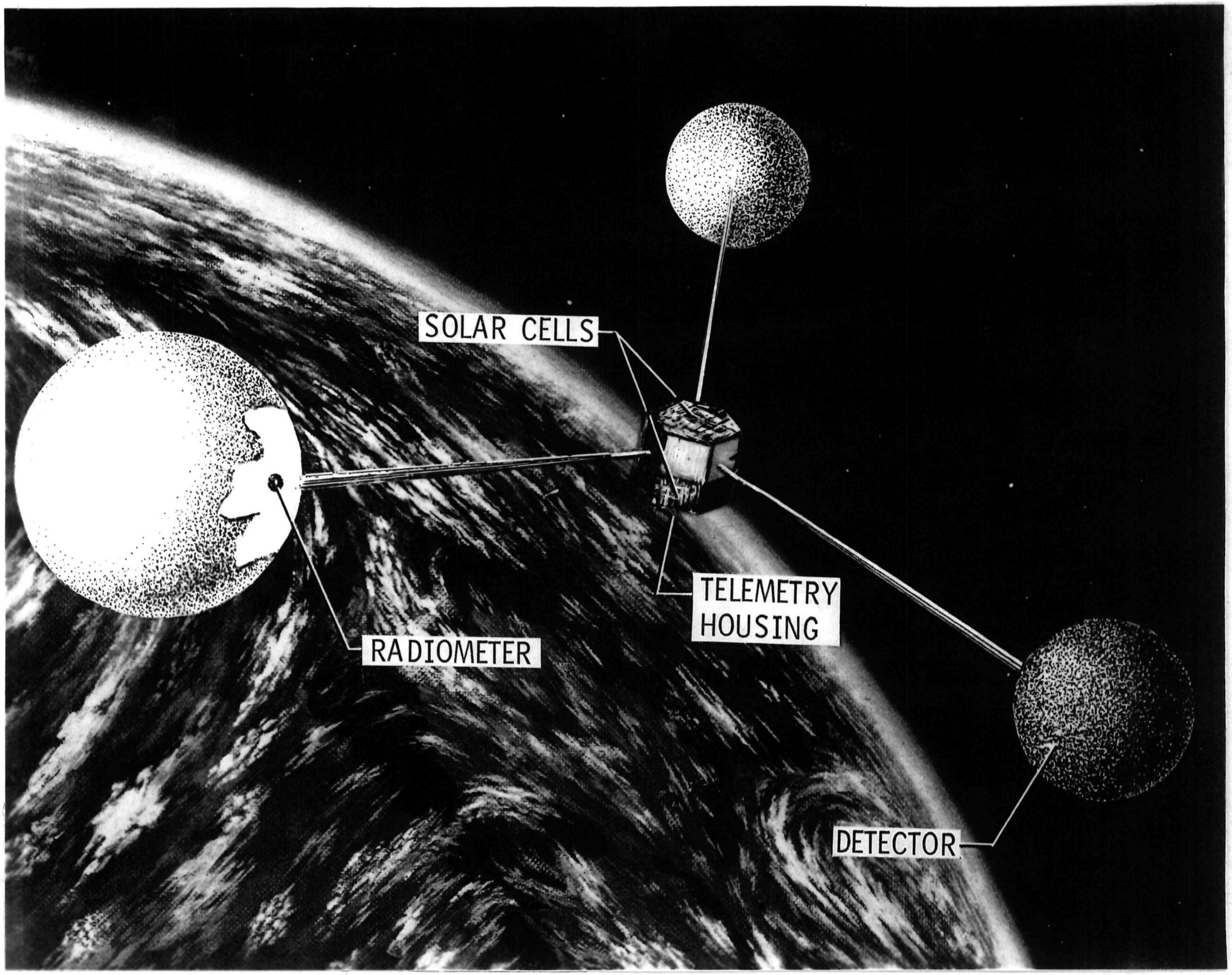

FIGURE 1. LONG-TERM ZONAL EARTH ENERGY BUDGET EXPERIMENT

The six elastic displacements are represented by one degree of freedom each, so that the spacecraft is simulated by a nine-degree-of-freedom system. For the purpose of numerical integration, the nine second-order Lagrangian equations have been transformed into eighteen first-order equations for the state variables, namely, the spacecraft generalized displacements and velocities. The first-order equations have been integrated numerically by a Runge-Kutta procedure. Note that the differential equations are highly nonlinear, so that no closed-form solution is possible.

The behavior of the spacecraft has been investigated for various cases, depending on the initial conditions. Computer results indicate that if the spacecraft is injected into orbit with zero initial displacements and velocities relative to an inertial space, then the sun's radiation and gravitational torques impart to the spacecraft a rotational motion about its own center that can be regarded as random. The results over a 20 orbit time interval do not show any tendency of the spacecraft to settle into an equilibrium corresponding to planar gravity-gradient stabilization. On the other hand, if the spacecraft is captured in the above equilibrium configuration, then it shows no tendency to escape this equilibrium state, but continues to oscillate slowly about that equilibrium. Hence, there is the possibility of capture in that equilibrium state. Note that the boom flexibility does not change the nominal configuration of the spacecraft significantly, and the thermal effects are not likely to change it appreciably either. Although a circular orbit was assumed, it should be pointed out that orbit eccentricity has a destabilizing influence on the gravity-gradient equilibrium state.

If the possibility of capture in planar gravity-gradient stabilization equilibrium is to be absolutely prevented, then a slightly different spacecraft design may deserve consideration. Indeed, a spacecraft with four equal booms instead of three, so that the ballons lie at the four corners of a regular

pyramid and the hub lies at the center of the pyramid, possesses sufficient spherical inertial symmetry to virtually eliminate gravity torques. This should permit the sun's radiation pressure to impart to the spacecraft a rotational motion that, for all practical purposes, can be considered as being random.

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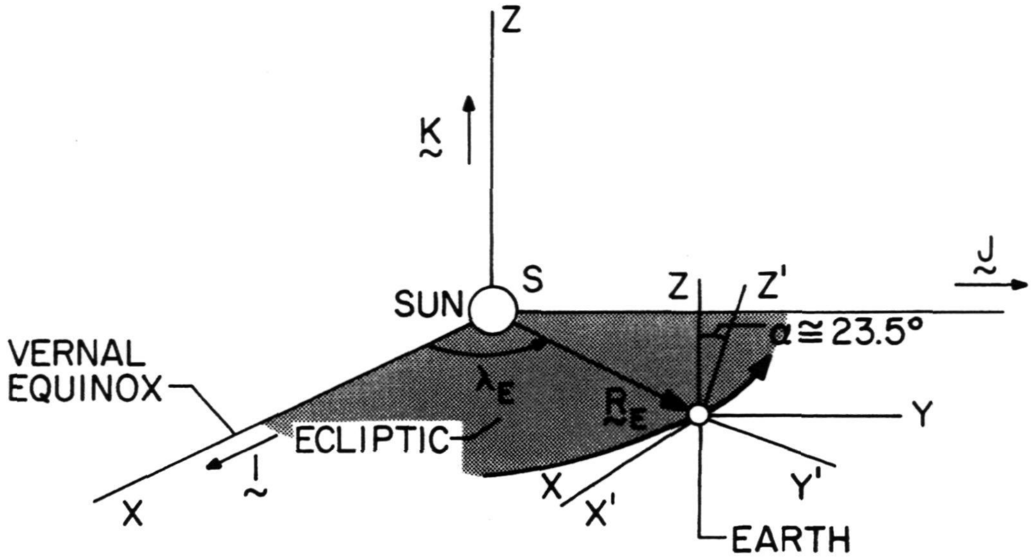

FIGURE 2. INERTIAL SPACE

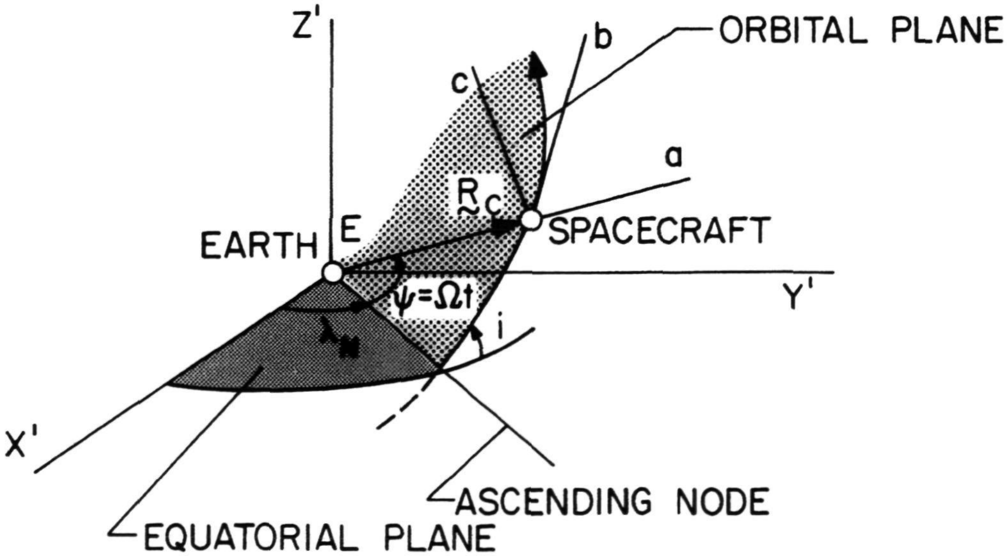

FIGURE 3. ORBITING SYSTEM

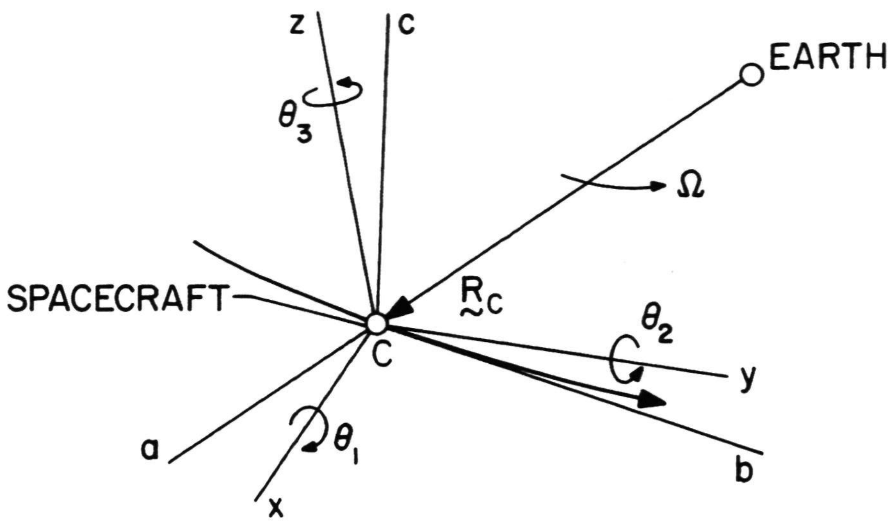

FIGURE 4. - ANGULAR COORDINATES

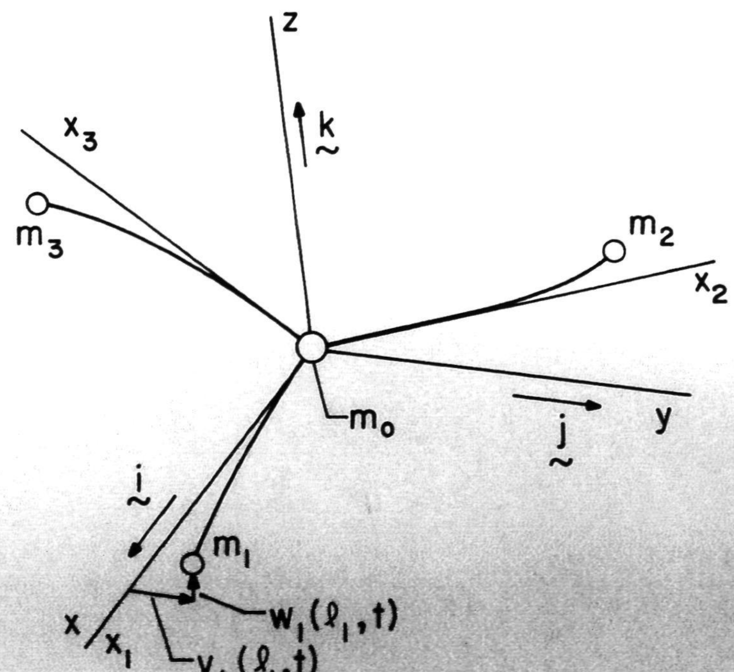

FIGURE 5. ELASTIC DISPLACEMENTS

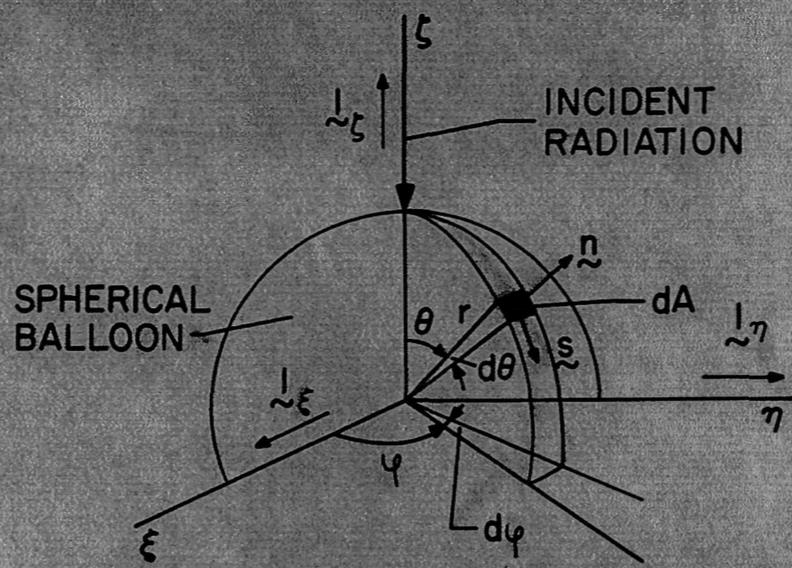

FIGURE 6. RADIATION FORCES ON BALLOON

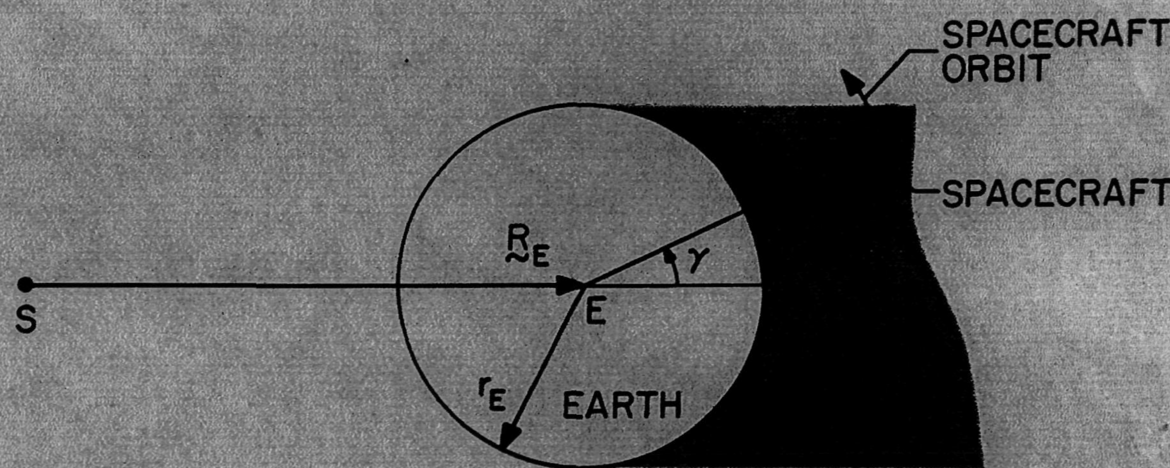

FIGURE 7. SPACECRAFT IN EARTH'S SHADOW